Journal of Computational and Applied Mathematics 108 (1999) 31–40

JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS

A relation between the kth derivate of the Dirac delta in $(P \pm i0)$ and the residue of distributions $(P \pm i0)^\lambda$

Manuel Antonio Aguirre Téllez *, Ana Lucía Barrenechea

Facultad de Ciencias Exactas, Depto. de Matematica, University de Buenos Aires, Pinto 399, 7000 Tandil, Argentina

Received 24 July 1998; received in revised form 3 March 1999

Abstract

The purpose of this paper is to give sense to the residue of the distribution $(P \pm i0)^\lambda$, when $\lambda = -n/2 - k$, $k = 0, 1, 2, \ldots,$ under certain conditions of n, p and q. This will be posible using a relation between the distribution $\delta^{(k)}(P \pm i0)$ and the residue of $(P \pm i0)^\lambda$. © 1999 Elsevier Science B.V. All rights reserved.

MSC: 46

Keywords: Theory of distributions

1. Introduction

Let $x = (x_1, \ldots, x_n)$ be a point of the n-dimensional Euclidean Space $\mathbb{R}^n$. Consider a nondegenate quadratic form in $\mathbb{R}^n$:

$$P = P(x) = x_1 + \cdots + x_p - x_{p+1} - \cdots - x_{p+q}, \tag{1}$$

where $p + q = n$.

The distributions $(P \pm i0)^\lambda$ are defined by

$$(P \pm i0)^\lambda = \lim_{\varepsilon \to 0} (P \pm i\varepsilon|x|^2)^\lambda, \tag{2}$$

where $\varepsilon > 0$, λ is a complex number and $|x|^2 = x_1^2 + \cdots + x_n^2$.

From [4], formulae (2) and (2′), we have

$$(P \pm i0)^\lambda = P_+^\lambda + e^{\pm\pi i\lambda} P_-^\lambda, \tag{3}$$

* Corresponding author. Tel.: +54-2293-44430; fax: +54-2293-47104/429234.

E-mail address: maguirre@exa.unicen.edu.ar (M.A.A. Téllez)

PII: S0377-0427(99)00098-9

where

$$P_+^{\lambda} = \begin{cases} P^{\lambda} & \text{if } P \geqslant 0, \\ 0 & \text{if } P < 0 \end{cases} \tag{4}$$

and

$$P_-^{\lambda} = \begin{cases} 0 & \text{if } P > 0, \\ (-P)^{\lambda} & \text{if } P \leqslant 0. \end{cases} \tag{5}$$

The distributions $(P \pm \mathrm{i}0)^{\lambda}$, considered in (2) is an holomorphic (distribution-valued) function of λ, everywhere except in points of the form $\lambda = -(n/2) - k$, where k is a nonnegative integer. Such points are first-order poles of this distributions and ([4, p. 276]):

$$\operatorname*{res}_{\lambda=-(n/2)-k} (P \pm \mathrm{i}0)^{\lambda} = \frac{\mathrm{e}^{\pm i(q/2)\pi} \pi^{n/2}}{4^k k! \Gamma(n/2+k)} L^k \delta, \tag{6}$$

where

$$L = \frac{\partial^2}{\partial x_1^2} + \cdots + \frac{\partial^2}{\partial x_p^2} - \cdots - \frac{\partial^2}{\partial x_{p+1}^2} - \cdots - \frac{\partial^2}{\partial x_{p+q}^2}. \tag{7}$$

Gel'fand and Shilov did not study the special case of λ being a negative integer, which is a simple pole of the distribution, in detail. This has been done by Jager and Bresters [3].

The purpose of this paper is to give relationships between the distribution $\delta^{(k)}((P \pm \mathrm{i}0))$ ([1, p. 344]) and the distribution $(P \pm \mathrm{i}0)^{\lambda}$ under certain conditions when $\lambda = -(n/2) - k$.

2. The distributions P_+^{λ} and P_-^{λ}

In this section, we will study different ways to express P_+^{λ} and P_-^{λ} in the distributional sense.

We know, from [4, p. 254, Eq. (10)], that the following formulaes are valid:

$$(P_+^{\lambda}, \varphi) = \frac{1}{4} \int_0^{\infty} u^{\lambda+((p+q)/2)-1} \left(\int_0^1 (1-t)^{\lambda} t^{(q-2)/2} \psi_1(u, tu)\, \mathrm{d}t \right) \mathrm{d}u \tag{8}$$

and from Gel'fand and Shilvov [4, p. 269, Eq. (47)]

$$(P_-^{\lambda}, \varphi) = \int_{P<0} (-P(x))^{\lambda} \varphi(x)\, \mathrm{d}x. \tag{9}$$

Now, making the bipolar change of variable (see [4, p. 253])

$$x_1 = r\omega_1, \ldots, x_p = r\omega_p, \qquad x_{p+1} = s\omega_{p+1}, \ldots, x_{p+q} = s\omega_{p+q}. \tag{10}$$

we obtain

$$(P_-^{\lambda}, \varphi) = \int_0^{\infty} \int_0^s (-1)^{\lambda} (r^2 - s^2)^{\lambda} r^{p-1} s^{q-1} \psi(r, s)\, \mathrm{d}r\, \mathrm{d}s, \tag{11}$$

where

$$\psi(r, s) = \int \varphi\, \mathrm{d}\Omega_p\, \mathrm{d}\Omega_q. \tag{12}$$

Putting $u=r^2$ and $v=s^2$ we obtain

$$(P_-^\lambda,\varphi)=(-1)^\lambda\int_0^\infty\int_0^v(u-v)^\lambda u^{(p-1)/2}v^{(q-1)/2}\psi_1(u,v)\frac{\mathrm{d}u}{2u^{1/2}}\frac{\mathrm{d}v}{2v^{1/2}},\tag{13}$$

where

$$\psi_1(u,v)=\psi(r,s)\tag{14}$$

so

$$(P_-^\lambda,\varphi)=\frac{(-1)^\lambda}{4}\int_0^\infty\int_0^v(u-v)^\lambda u^{(p-2)/2}v^{(q-2)/2}\psi_1(u,v)\,\mathrm{d}u\,\mathrm{d}v.\tag{15}$$

Making the change of variable $u=tv$, the last expression becomes

$$(P_-^\lambda,\varphi)=\frac{1}{4}\int_0^\infty v^{\lambda+(p+q-2)/2}\left(\int_0^1(1-t)^\lambda t^{(p-2)/2}\psi_1(tv,v)\,\mathrm{d}t\right)\mathrm{d}v.\tag{16}$$

Similarly, from the above formulae, (16) can be written in the following form:

$$(P_-^\lambda,\varphi)=\frac{(-1)^{\lambda+1}}{4}\int_0^\infty u^{\lambda+(p+q-2)/2}\left(\int_0^1(1-t)^\lambda t^{(q-2)/2}\psi_1(u,tu)\,\mathrm{d}t\right)\mathrm{d}u.\tag{17}$$

From (8), (16) and (17), we have

$$(P_+^\lambda,\varphi)=\int_0^\infty u^{\lambda+((p+q)/2)-1}\,{}_q\Phi_\lambda(u)\,\mathrm{d}u\tag{18}$$

and

$$(P_-^\lambda,\varphi)=\int_0^\infty u^{\lambda+((p+q)/2)-1}\,{}_p\Phi_\lambda(u)\,\mathrm{d}u,\tag{19}$$

where

$${}_q\Phi_\lambda(u)=\frac{1}{4}\int_0^1(1-t)^\lambda t^{(q-2)/2}\psi_1(u,tu)\,\mathrm{d}t\tag{20}$$

and

$${}_p\Phi_\lambda(u)=\frac{1}{4}\int_0^1(1-t)^\lambda t^{(p-2)/2}\psi_1(u,tu)\,\mathrm{d}t.\tag{21}$$

Using (17) and (20) we have

$$(P_-^\lambda,\varphi)=(-1)^{\lambda+1}\int_0^\infty u^{\lambda+((p+q)/2)-1}\,{}_q\Phi_\lambda(u)\,\mathrm{d}u.\tag{22}$$

The distributions P_+^λ and P_-^λ have singularities in points of the form $\lambda=-(n/2)-k$ or $\lambda=-k$, $k=0,1,2,\dots$. In general, from Gel'fand and Shilov [4, p. 278], we know that

$$\delta^{(k)}(P_+)=(-1)^k k!\operatorname*{res}_{\lambda=-k-1}P_+^\lambda\tag{23}$$

and similarly

$$\delta^{(k)}(P_-)=(-1)^k k!\operatorname*{res}_{\lambda=-k-1}P_-^\lambda.\tag{24}$$

As the kind of singularity of λ depends if the number n is odd or even, so if p and q are odd and even. We can classify them as follows:

CaseP_+^λ:

♯ *When p is odd and q is even*: $\lambda=-k$, and $\lambda=-(n/2)-k$, $k=0,1,2,\dots$ are simple poles. So we have

$$\operatorname*{res}_{\lambda=-k} P_+^\lambda = \frac{(-1)^{k-1}\delta_1^{(k-1)}(P)}{(k-1)!} \quad [4,\ \text{p. } 352], \tag{25}$$

and

$$\operatorname*{res}_{\lambda=-n/2-k} P_+^\lambda = \frac{(-1)^{q/2}\pi^{n/2}}{4^k\Gamma(n/2+k)k!}L^k\delta(x) \quad [4,\ \text{p. } 352]. \tag{26}$$

♯ *When p is even and q is odd*: $\lambda=-k$, $k=0,1,2,\dots,$ are simple poles. So we have

$$\operatorname*{res}_{\lambda=-k} P_+^\lambda = \frac{(-1)^{k-1}\delta_1^{(k-1)}(P)}{(k-1)!} \quad [4,\ \text{p. } 352]. \tag{27}$$

In this case, we observe from [4, p. 260], that P_+^λ is regular in $\lambda=-(n/2)-k$; $k=0,1,2,3,\dots$. Therefore

$$\operatorname*{res}_{\lambda=-(n/2)-k} P_+^\lambda = 0. \tag{28}$$

♯ *When p is odd and q is even*: $\lambda=-k$, and $\lambda=-(n/2)-k$ are simple poles with

$$\operatorname*{res}_{\lambda=-k} P_+^\lambda = \frac{(-1)^{k-1}\delta_1^{(k-1)}(P)}{(k-1)!} \tag{29}$$

and

$$\operatorname*{res}_{\lambda=-(n/2)-k} P_+^\lambda = \frac{(-1)^{q/2}\pi^{(n/2)}}{4^k k!\Gamma((n/2)+k)}L^k\delta(x). \tag{30}$$

♯ *When p is even and q is even*: $\lambda=-(n/2)-k$, $k=0,1,2,\dots$ are simple poles, but we use different expressions for the residue

$$\operatorname*{res}_{\lambda=-k} P_+^\lambda = \frac{(-1)^{k-1}\delta_1^{(k-1)}(P)}{(k-1)!}, \quad k<\frac{n}{2} \tag{31}$$

and

$$\operatorname*{res}_{\lambda=-(n/2)-k} P_+^\lambda = \frac{(-1)^{(n/2)+k-1}\delta_1^{((n/2)+k-1)}(P)}{\Gamma((n/2)+k)} + \frac{(-1)^{q/2}\pi^{(n/2)}}{4^k k!\Gamma((n/2)+k)}L^k\delta(x). \tag{32}$$

♯ *When p and q are both odd*: $\lambda=-k$ $(k<(n/2))$ are simple poles; and $\lambda=-(n/2)-k$, $k=1,2,\dots,$ are poles of order 2. So

$$\operatorname*{res}_{\lambda=-k} P_+^\lambda = \frac{(-1)^{k-1}\delta_1^{(k-1)}(P)}{(k-1)!}, \quad k\geqslant\frac{n}{2} \tag{33}$$

and in the case of the poles of order 2 we have

$$c_{-1}^{(k)} = \frac{(-1)^{(n/2)+k-1}\delta_1^{(n/2)+k-1}(P)}{\Gamma((n/2)+k)} + \frac{(-1)^{(q+1)/2}\pi^{(n/2)}}{4^k k!\Gamma((n/2)+k)}\left[\Psi\left(\frac{p}{2}\right)-\Psi\left(\frac{n}{2}\right)\right]L^k\delta(x) \tag{34}$$

and

$$c_{-2}^{(k)} = \frac{(-1)^{(q+1)/2}\pi^{(n/2)-1}}{4^k k!\Gamma((n/2)+k)} L^k \delta(x), \tag{35}$$

where $c_{-1}^{(k)}$ and $c_{-2}^{(k)}$ are the coefficients of the Laurent series of the holomorphic functions P_+^λ developed in those points. And

$$\Psi(x) = \frac{\Gamma'(x)}{\Gamma(x)} \tag{36}$$

and

$$\Psi(k) = -C + 1 + \frac{1}{2} + \cdots + \frac{1}{k-1},$$

where C is the Euler's constant.

Case P_-^λ.

♯ *When* p *is even and* q *is odd*: $\lambda = -(n/2) - k$ and $\lambda = -k$, $k = 0, 1, 2, \ldots$, are simple poles; so

$$\operatorname*{res}_{\lambda=-(n/2)-k} P_-^\lambda = \frac{(-1)^{(p/2)+k}\pi^{(n/2)}}{4^k k!\Gamma((n/2)+k)} L^k \delta(x) \tag{37}$$

and

$$\operatorname*{res}_{\lambda=-k} P_-^\lambda = \frac{\delta_2^{(k-1)}(P)}{(k-1)!} = \frac{(-1)^{k-1}}{(k-1)!}\delta_1^{(k-1)}(-P), \tag{38}$$

where we have used the relations between $\delta_1^{(k-1)}(-P)$ and $\delta_2^{(k-1)}(P)$, see [4, p. 251] in fact

$$\delta_1^{(k-1)}(-P) = (-1)^{k-1}\delta_2^{(k-1)}(P). \tag{39}$$

♯ *When* p *is odd and* q *is even*: $\lambda = -k$ are simple poles, and

$$\operatorname*{res}_{\lambda=-k} P_-^\lambda = \frac{\delta_1^{(k-1)}(-P)}{(k-1)!}. \tag{40}$$

In this case, $\lambda = -(n/2) - k$, $k = 0, 1, 2, \ldots$ are regular points as in the case of P_+^λ, so

$$\operatorname*{res}_{\lambda=-(n/2)-k} P_-^\lambda = 0 \tag{41}$$

♯ *When* p *and* q *are both even*: $\lambda = -k$, $k = 0, 1, 2, \ldots$ are simple poles; but

$$\operatorname*{res}_{\lambda=-k} P_-^\lambda = \frac{\delta_1^{(k-1)}(-P)}{(k-1)!}, \quad k < \frac{n}{2} \tag{42}$$

and

$$\operatorname*{res}_{\lambda=-k} P_-^\lambda = \frac{\delta_2^{((n/2)+k-1)}(P)}{\Gamma((n/2)+k)} + \frac{(-1)^{(p/2)+k}\pi^{(n/2)}}{4^k k!\Gamma((n/2)+k)} L^k \delta(x), \quad k \geqslant \frac{n}{2}. \tag{43}$$

♯ *When* p *and* q *are both odd*: $\lambda = -1, -2, \ldots, -(n/2) - k$, are simple poles when $k < (n/2)$, and $\lambda = -(n/2) - k, k = 1, 2, \ldots$ are poles of order 2, so

$$\operatorname*{res}_{\lambda=-k} P_-^\lambda = \frac{\delta_2^{(k-1)}(P)}{(k-1)!} = \frac{(-1)^{k-1}\delta_1^{(k-1)}(-P)}{(k-1)!} \tag{44}$$

and

$$c_{-1}^{(k)} = \frac{\delta_2^{((n/2)+k-1)}(P)}{\Gamma((n/2)+k)} + \frac{(-1)^{((p+1)/2)+k}\pi^{(n/2)}}{4^k k!\Gamma((n/2)+k)}\left[\Psi\left(\frac{q}{2}\right) - \Psi\left(\frac{n}{2}\right)\right] L^k\delta(x), \tag{45}$$

when $\Psi(x)$ was defined in (36) and

$$c_{-2}^{(k)} = \frac{(-1)^{1/2(p+1)+1+k}\pi^{(n/2)-1}}{4^k k!\Gamma((n/2)+k)} L^k\delta(x). \tag{46}$$

3. The distributions $(P \pm \mathrm{i}0)^\lambda$

The aim of this section is to give an integral representation of the distributions $(P \pm \mathrm{i}0)^\lambda$. Those distributions were defined by expression (2).

Using formulaes (3), (18) and (19) we can write

$$\langle (P \pm \mathrm{i}0)^\lambda, \varphi\rangle = \int_0^\infty u^{\lambda+((p+q)/2)-1}\, {}_qF_p(u)\,\mathrm{d}u, \tag{47}$$

where

$${}_qF_p(\lambda, u) = ({}_q\Phi_\lambda(u) + \mathrm{e}^{\lambda\pi\mathrm{i}}\, {}_p\Phi_\lambda(u)). \tag{48}$$

Finally, using (17), we can write

$$\langle (P \pm \mathrm{i}0)^\lambda, \varphi\rangle = c_\lambda \int_0^\infty u^{\lambda+((p+q)/2)-1}\, {}_q\Phi_\lambda(u)\,\mathrm{d}u, \tag{49}$$

where

$$c_\lambda = [(-1)^{\lambda+1} + \mathrm{e}^{\pm\mathrm{i}\lambda\pi}]. \tag{50}$$

Using formulae (49), we are able to prove that the residium of $(P \pm \mathrm{i}0)^\lambda$ in the points of the form $\lambda = -k$, $k = 0, 1, 2, \ldots$ is zero. In fact, from (20), ${}_q\Phi_\lambda(\mathrm{u})$ has a simple point in $\lambda = -k$, so we can develop in Laurent series the function in a neighborhood of $\lambda = -k$ so that we can write

$${}_q\Phi_\lambda(u) = \frac{\Phi_0(u)}{\lambda + k} + \Phi_1(\lambda, u), \tag{51}$$

where

$$\Phi_0 = \operatorname*{res}_{\lambda=-k}\, {}_q\Phi_\lambda(u) = \lim_{\lambda\to -k}(\lambda + k)\, {}_q\Phi_\lambda(u)$$

and $\Phi_1(\lambda, u)$ is a regular function in $\lambda = -k$.

Inserting (51) in (49) we have

$$\begin{aligned}\langle (P \pm \mathrm{i}0)^\lambda, \varphi\rangle &= c_\lambda \int_0^\infty u^{\lambda+((p+q)/2)-1}\left(\frac{\Phi_0(u)}{\lambda + k} + \Phi_1(\lambda, u)\right)\mathrm{d}u,\\ \langle (P \pm \mathrm{i}0)^\lambda, \varphi\rangle &= \frac{c_\lambda}{\lambda + k}\int_0^\infty u^{\lambda+((p+q)/2)-1}\Phi_0(u)\,\mathrm{d}u + c_\lambda \int_0^\infty u^{\lambda+((p+q)/2)-1}\Phi_1(\lambda, u)\,\mathrm{d}u.\end{aligned} \tag{52}$$

When n is odd or n is even but $k < (n/2)$, from (52) we obtain

$$\begin{aligned}\operatorname*{res}_{\lambda=-k}\langle (P\pm \mathrm{i}0)^{\lambda},\varphi\rangle &= \lim_{\lambda\to -k}\langle (\lambda+k)(P\pm \mathrm{i}0)^{\lambda},\varphi\rangle\\ &= \lim_{\lambda\to -k}\left(c_{\lambda}\int_0^{\infty} u^{\lambda+((p+q)/2)-1}\Phi_0(u)\,\mathrm{d}u\right)\\ &\quad + \lim_{\lambda\to -k}(\lambda+k)c_{\lambda}\int_0^{\infty} u^{\lambda+((p+q)/2)-1}\Phi_1(\lambda,u)\,\mathrm{d}u\\ &= 0.\end{aligned}$$

Therefore,

$$\operatorname*{res}_{\lambda=-k}\langle (P\pm \mathrm{i}0)^{\lambda},\varphi\rangle = 0 \quad \text{when } n \text{ is odd or } n \text{ is even but } k<\frac{n}{2}. \tag{53}$$

Notice that this property appears in [4, p. 278].

In the next section we will study the residue of $(P\pm \mathrm{i}0)^{\lambda}$ in $\lambda = -(n/2)-k$, $k=0,1,2,\ldots$. We are going to consider two cases: n odd and n even.

Case 1: *n is odd.*

1.1. p is odd and q is even: From (47) and taking into account formulae (48),(18) and (19), we have

$$\begin{aligned}\operatorname*{res}_{\lambda=-(n/2)-k}\langle (P\pm \mathrm{i}0)^{\lambda},\varphi\rangle &= \operatorname*{res}_{\lambda=-(n/2)-k}\frac{1}{4}\int_0^{\infty} u^{\lambda+(n/2)-1}\,{}_q\Phi_{\lambda}(u)\,\mathrm{d}u\\ &\quad + \operatorname*{res}_{\lambda=-(n/2)-k}\mathrm{e}^{\pm\lambda\pi\mathrm{i}}\frac{(-1)^{\lambda+1}}{4}\int_0^{\infty} u^{\lambda+(n/2)-1}\,{}_p\Phi_{\lambda}(u)\,\mathrm{d}u\\ &= \operatorname*{res}_{\lambda=-(n/2)-k}\langle P_+^{\lambda},\varphi\rangle + \mathrm{e}^{\pm\mathrm{i}\pi(-(n/2)-k)}\operatorname*{res}_{\lambda=-(n/2)-k}\langle P_-^{\lambda},\varphi\rangle.\end{aligned}$$

Now, using (30) and (41), we have

$$\operatorname*{res}_{\lambda=-(n/2)-k}\langle (P\pm \mathrm{i}0)^{\lambda},\varphi\rangle = a_{k,n}(-1)^{q/2}L^k\delta, \tag{54}$$

where

$$a_{k,n} = \frac{\pi^{(n/2)}}{2^{2k}k!\,\Gamma((n/2)+k)} \tag{55}$$

1.2. p is even and q is odd: From (47) and taking into account formulae (48),(28), (18) and (19), we have

$$\begin{aligned}\operatorname*{res}_{\lambda=-(n/2)-k}\langle (P\pm \mathrm{i}0)^{\lambda},\varphi\rangle &= \mathrm{e}^{\pm\pi\mathrm{i}(-(n/2)-k)}\operatorname*{res}_{\lambda=-(n/2)-k}\langle P_-^{\lambda},\varphi\rangle\\ &= \mathrm{e}^{\pm\pi\mathrm{i}(-(n/2)-k)}a_{k,n}(-1)^{p/2}(-1)^k\langle L^k\delta,\varphi\rangle\\ &= \mathrm{e}^{\pm(n/2)\pi\mathrm{i}}(-1)^{p/2}a_{k,n}\langle L^k\delta,\varphi\rangle\\ &= \mp\mathrm{i}(-1)^{(q-1)/2}a_{k,n}\langle L^k\delta,\varphi\rangle.\end{aligned}$$

So

$$\operatorname*{res}_{\lambda=-(n/2)-k}\langle (P\pm \mathrm{i}0)^{\lambda},\varphi\rangle = \pm\mathrm{e}^{\pm q\pi\mathrm{i}/2}a_{k,n}\langle L^k\delta,\varphi\rangle,$$

where $a_{k,n}$ was defined in (55).

Summarizing,

1. If p is odd and q is even,
$$\operatorname*{res}_{\lambda=-(n/2)-k}\langle(P\pm \mathrm{i}0)^{\lambda},\varphi\rangle = \mathrm{e}^{\pm q\pi\mathrm{i}/2}a_{k,n}\langle L^k\delta,\varphi\rangle. \tag{56}$$
2. If p is even and q is odd,
$$\operatorname*{res}_{\lambda=-(n/2)-k}\langle(P\pm \mathrm{i}0)^{\lambda},\varphi\rangle = \pm\mathrm{e}^{\pm q\pi\mathrm{i}/2}a_{k,n}\langle L^k\delta,\varphi\rangle. \tag{57}$$

Case 2: *n is even.*

2.1. p and q are both even: From (47) and considering (32) and (43), we have

$$\begin{aligned}
\operatorname*{res}_{\lambda=-(n/2)-k}\langle(P\pm \mathrm{i}0)^{\lambda},\varphi\rangle &= \operatorname*{res}_{\lambda=-(n/2)-k}\langle P_+^{\lambda},\varphi\rangle + \mathrm{e}^{(-(n/2)-k)\pi\mathrm{i}}\operatorname*{res}_{\lambda=-(n/2)-k}\langle P_-^{\lambda},\varphi\rangle \\
&= \Bigg\langle \frac{(-1)^{(n/2)+k-1}}{\Gamma((n/2)+k)}\delta_1^{((n/2)+k-1)}(P) + \frac{(-1)^{(n/2)+k}}{\Gamma((n/2)+k)}\delta_2^{((n/2)+k-1)}(P) \\
&\quad + \frac{(-1)^{(n/2)+k+q/2}\pi^{(n/2)}}{2^{2k}k!\Gamma((n/2)+k)}L^k\delta + (-1)^{(n/2)+k}\frac{(-1)^{p/2}\pi^{(n/2)}(-1)^k}{2^{2k}k!\Gamma((n/2)+k)}L^k\delta,\varphi\Bigg\rangle \\
&= \Bigg\langle \frac{(-1)^{(n/2)+k}}{\Gamma((n/2)+k)}[-\delta_1^{((n/2)+k-1)}(P)+\delta_2^{((n/2)+k-1)}(P)] \\
&\quad + \frac{2(-1)^{q/2}}{2^{2k}k!\Gamma((n/2)+k)}L^k\delta,\varphi\Bigg\rangle.
\end{aligned} \tag{58}$$

On the other hand, in [2, p. 11], formula (II, 1, 5) had been proved that
$$\delta_2^{(k)}(P)-\delta_1^{(k)}(P) = (-1)^k a_{q,n,k}L^{k-(n/2)+1}\delta, \tag{59}$$
where
$$a_{q,n,k} = \frac{(-1)^{q/2}\pi^{(n/2)}}{4^{k+1-(n/2)}(k+1-(n/2))!}.$$
Therefore
$$\operatorname*{res}_{\lambda=-(n/2)-k}\langle(P\pm \mathrm{i}0)^{\lambda},\varphi\rangle = \left\langle \frac{(-1)^{q/2}\pi^{(n/2)}}{2^{2k}k!\Gamma((n/2)+k)}L^k\delta,\varphi\right\rangle. \tag{60}$$

2.2. p and q are both odd: Using again (47), (32) and (43), we have

$$\begin{aligned}
&= \Bigg\langle \frac{(-1)^{(n/2)+k-1}}{\Gamma((n/2)+k)}\delta_1^{((n/2)+k-1)}(-P) + \frac{(-1)^{(q+1)/2}\pi^{(n/2)-1}}{2^{2k}k!\Gamma((n/2)+k)}\left[\Psi\left(\frac{p}{2}\right)-\Psi\left(\frac{n}{2}\right)\right]L^k\delta \\
&\quad +(-1)^{(n/2)+k}\Bigg\{\frac{(-1)^{(n/2)+k+q/2}}{\Gamma((n/2)+k)}\delta_1^{((n/2)+k-1)}(-P) \\
&\quad + \frac{(-1)^{(q+1)/2}\pi^{(n/2)-1}}{2^{2k}k!\Gamma((n/2)+k)}\left[\Psi\left(\frac{q}{2}\right)-\Psi\left(\frac{n}{2}\right)\right]\Bigg\}(-L)^k\delta,\varphi\Bigg\rangle
\end{aligned}$$

$$= \left\langle \frac{(-1)^{(n/2)+k}}{\Gamma((n/2)+k)}[\delta_2^{((n/2)+k-1)}(P) - \delta_1^{((n/2)+k-1)}(P)] \right.$$

$$\left. + \frac{(-1)^{(q+1)/2}\pi^{(n/2)-1}}{2^{2k}k!\Gamma((n/2)+k)}\left[\Psi\left(\frac{q}{2}\right) - \Psi\left(\frac{p}{2}\right)\right] L^k\delta, \varphi \right\rangle.$$

Taking into account formulae (II,1,8), from Aguirre Téllez [2, p. 11], in fact

$$\delta_2^{(k)}(P) - \delta_1^{(k)}(P) = (-1)^k b_{k,q,n}\left[\Psi\left(\frac{p}{2}\right) - \Psi\left(\frac{q}{2}\right)\right] L^{k-(n/2)+1}\delta,$$

where

$$b_{k,n,q} = \frac{(-1)^{1/2(q+1)}\pi^{(n/2)-1}}{4^{k-(n/2)+1}(k-(n/2)+1)!}$$

we have

$$\operatorname*{res}_{\lambda=-(n/2)-k}\langle (P\pm \mathrm{i}0)^\lambda, \varphi\rangle = \left\langle \frac{(-1)^{(n/2)+k}}{\Gamma((n/2)+k)}\left\{\frac{(-1)^{(n/2)+k-1}(-1)^{(q+1)/2}\pi^{(n/2)-1}}{4^{(n/2)-1-(n/2)+1+k}((n/2)+k-1-(n/2)+1)!}\right.\right.$$

$$\left.\left.\times\left[\Psi\left(\frac{p}{2}\right) - \Psi\left(\frac{q}{2}\right)\right]\right\} L^{(n/2)+k-1-(n/2)+1}\delta, \varphi\right\rangle$$

$$= \left\langle \frac{(-1)(-1)^{(q+1)/2}\pi^{(n/2)-1}}{\Gamma((n/2)+k)2^{2k}k!}\left[\Psi\left(\frac{p}{2}\right) - \Psi\left(\frac{q}{2}\right)\right] L^k\delta \right.$$

$$\left. + \frac{(-1)^{(q+1)/2}\pi^{(n/2)-1}}{\Gamma((n/2)+k)2^{2k}k!}\left[\Psi\left(\frac{p}{2}\right) - \Psi\left(\frac{q}{2}\right)\right] L^k\delta, \varphi \right\rangle$$

$$= 0.$$

4. The main result

In this section we are going to give sense to

$$\operatorname*{res}_{\lambda=-k-1}(P\pm \mathrm{i}0)^\lambda$$

where $k \geqslant (n/2)-1$, $k = 0,1,2,\ldots$ and n, p and q are all even.

To obtain the result we need the following formulae:

$$\delta^{(k)}(P\pm \mathrm{i}0) = \{\delta^{(k)}(P_+) + \mathrm{e}^{\pm\pi \mathrm{i}(k+1)}\delta^{(k)}(P_-)\} \quad [1,\ \text{p. 345, formulae (4.5)}]. \tag{61}$$

Theorem 1. *Let k be a nonnegative integer such that $k \geqslant (n/2)$, where n is the dimension of the space. If n is even, the following formula is valid*:

$$\operatorname*{res}_{\lambda=-k-1}(P\pm \mathrm{i}0)^\lambda = \frac{(-1)^k}{k!}\delta^{(k)}(P\pm \mathrm{i}0), \tag{62}$$

when p and q are both even.

Proof. Taking into account (3), we can write

$$\operatorname*{res}_{\lambda=-(n/2)-h}\langle (P\pm \mathrm{i}0)^{\lambda},\varphi\rangle = \operatorname*{res}_{\lambda=-(n/2)-h}\langle P_{+}^{\lambda},\varphi\rangle + \mathrm{e}^{\lambda\pi \mathrm{i}}|_{\lambda=-(n/2)-h}\operatorname*{res}_{\lambda=-(n/2)-h}\langle P_{-}^{\lambda},\varphi\rangle.$$

As $k\geqslant (n/2)-1$; $h=k-(n/2)+1$ is a nonnegative integer, and $\lambda=-k-1$. So we can rewrite the last expression as follows:

$$\operatorname*{res}_{\lambda=-k-1}\langle (P\pm \mathrm{i}0)^{\lambda},\varphi\rangle = \operatorname*{res}_{\lambda=-k-1}\langle P_{+}^{\lambda},\varphi\rangle + \mathrm{e}^{(-k-1)\pi \mathrm{i}}\operatorname*{res}_{\lambda=-k-1}\langle P_{-}^{\lambda},\varphi\rangle.$$

Using (60), (23) and (24), we have

$$\begin{aligned}(L^{k-(n/2)+1}\delta,\varphi) &= c_{k-(n/2)+1,n}^{-1}\operatorname*{res}_{\lambda=-k-1}\langle (P\pm \mathrm{i}0)^{\lambda},\varphi\rangle\\ &= c_{k-(n/2)+1,n}^{-1}\left[\frac{(-1)^{k}}{k!}\delta^{(k)}(P_{+})+\frac{(-1)^{k+1}}{k!}(-1)^{k}\delta^{(k)}(P_{-})\right]\\ &= c_{k-(n/2)+1,n}^{-1}\left[\frac{(-1)^{k}}{k!}\delta^{(k)}(P_{+})-\frac{1}{k!}\delta^{(k)}(P_{-})\right]\\ &= \frac{c_{k-(n/2)+1,n}^{-1}}{k!}[(-1)^{k}\delta^{(k)}(P_{+})-\delta^{(k)}(P_{-})],\end{aligned}$$

where

$$c_{k-(n/2)+1,n} = \frac{(-1)^{q/2}\pi^{(n/2)}}{2^{2(k-(n/2)+1)}(k-(n/2)+1)!\,\Gamma(k+1)}.$$

Applying formulae (61), we have

$$\begin{aligned}(L^{k-(n/2)+1}\delta,\varphi) &= c_{k-(n/2)+1,n}^{-1}\operatorname*{res}_{\lambda=-k-1}\langle (P\pm \mathrm{i}0)^{\lambda},\varphi\rangle\\ &= \frac{c_{k-(n/2)+1,n}^{-1}(-1)^{k}}{k!}\delta^{(k)}(P\pm \mathrm{i}0).\end{aligned}$$

Or equivalently we obtain (62)

$$\operatorname*{res}_{\lambda=-k-1}(P\pm \mathrm{i}0)^{\lambda} = \frac{(-1)^{k}}{k!}\delta^{(k)}(P\pm \mathrm{i}0). \qquad \square$$

References

[1] M. Aguirre Téllez, The Distribution $\delta^{(k)}(P\pm \mathrm{i}0-m^{2})$, J. Comput. Appl. Math. 88 (1997) 339–348.

[2] M. Aguirre Téllez, Relations of k-th derivative of Dirac delta in hypercone with ultrahyperbolic operator, to accept in Journal of Mathematics: Teoría y Aplicaciones, Vol. 6/2 (1999) Universidad de Costa Rica.

[3] D.N. Bresters, On distributions connected with quadratic forms, SIAM J. Appl. Math. 16 (1968) 563–581.

[4] I.M. Gel'fand, G.M. Shilov, Generalized Functions, Vol. I, Academic Press, New York, 1964.